

RELATIVE SINGULARITY CATEGORY OF A NON-COMMUTATIVE RESOLUTION OF SINGULARITIES

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ABSTRACT. In this article, we study a triangulated category associated with a non-commutative resolution of singularities. In particular, we give a complete description of this category in the case of a curve with nodal singularities, classifying its indecomposable objects and computing its Auslander–Reiten quiver and K-group.

1. INTRODUCTION

This article grew up from an attempt to generalize the following statement, which is a consequence of a theorem of Buchweitz [5, Theorem 4.4.1] and results on idempotent completions of triangulated categories [17, 22]. Let X be an algebraic variety with *isolated Gorenstein* singularities, $Z = \text{Sing}(X) = \{x_1, \dots, x_p\}$ and $\widehat{\mathcal{O}}_i := \widehat{\mathcal{O}}_{X, x_i}$ for all $1 \leq i \leq p$. Then we have an equivalence of triangulated categories

$$(1) \quad \left(\frac{D^b(\text{Coh}(X))}{\text{Perf}(X)} \right)^\omega \xrightarrow{\sim} \bigvee_{i=1}^p \underline{\text{MCM}}(\widehat{\mathcal{O}}_i).$$

The left-hand side of (1) stands for the idempotent completion of the Verdier quotient $D^b(\text{Coh}(X))/\text{Perf}(X)$ (known to be triangulated by [2]), whereas on the right-hand side $\underline{\text{MCM}}(\widehat{\mathcal{O}}_i)$ denotes the stable category of maximal Cohen-Macaulay modules over $\widehat{\mathcal{O}}_i$.

We want to generalize this construction as follows. Let $\mathcal{F}' \in \text{Coh}(X)$, $\mathcal{F} := \mathcal{O} \oplus \mathcal{F}'$ and $\mathcal{A} := \mathcal{E}nd_X(\mathcal{F})$. Consider the ringed space $\mathbb{X} := (X, \mathcal{A})$. It is well-known that the functor

$$\mathcal{F} \overset{\mathbb{L}}{\otimes}_X - : \text{Perf}(X) \longrightarrow D^b(\text{Coh}(\mathbb{X}))$$

is fully faithful, see for instance [7, Theorem 2]. If $\text{gl.dim}(\text{Coh}(\mathbb{X})) < \infty$ then \mathbb{X} can be viewed as a *non-commutative* (or *categorical*) resolution of singularities of X , in the spirit of works of Van den Bergh [27], Kuznetsov [18] and Lunts [19]. To measure the difference between $\text{Perf}(X)$ and $D^b(\text{Coh}(\mathbb{X}))$, we suggest to study the triangulated category

$$(2) \quad \Delta_X(\mathbb{X}) := \left(\frac{D^b(\text{Coh}(\mathbb{X}))}{\text{Perf}(X)} \right)^\omega,$$

which we shall call *relative singularity category*. Assuming \mathcal{F} to be locally free on $U := X \setminus Z$, we prove an analogue of the “localization equivalence” (1) for the category $\Delta_X(\mathbb{X})$. Using the negative K-theory of derived categories of Schlichting [23], we also describe the Grothendieck group of $\Delta_X(\mathbb{X})$.

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The main result of our article is a complete description of $\Delta_Y(\mathbb{Y})$ in case Y is an arbitrary *curve* with *nodal singularities* and $\mathcal{F}' := \mathcal{I}_Z$ is the ideal sheaf of the singular locus of Y . We show that $\Delta_Y(\mathbb{Y})$ splits into a union of p blocks: $\Delta_Y(\mathbb{Y}) \xrightarrow{\sim} \bigvee_{i=1}^p \Delta_i$, where p is the number of singular points of Y . Moreover, each block Δ_i turns out to be equivalent to the category Δ_{nd} defined as follows:

$$\Delta_{\text{nd}} := \frac{\text{Hot}^b(\text{pro}(A_{\text{nd}}))}{\text{Hot}^b(\text{add}(P_*))},$$

where A_{nd} is the completed path algebra of the following quiver with relations:

$$\begin{array}{c} \alpha \\ \curvearrowright \\ - \quad \quad \quad * \quad \quad \quad + \\ \curvearrowleft \\ \beta \end{array} \quad \begin{array}{c} \gamma \\ \curvearrowright \\ * \quad \quad \quad + \\ \curvearrowleft \\ \delta \end{array} \quad \delta\alpha = 0, \quad \beta\gamma = 0$$

and P_* is the indecomposable projective A_{nd} -module corresponding to the vertex $*$. We prove that the category Δ_{nd} is idempotent complete and \mathbf{Hom} -finite. Moreover, we give a complete classification of indecomposable objects of Δ_{nd} .

Finally, we show that Δ_{nd} has the following interesting description:

$$\Delta_{\text{nd}} \xrightarrow{\sim} \left(\frac{D^b(\Lambda - \text{mod})}{\text{Band}(\Lambda)} \right)^\omega,$$

where Λ is the path algebra of the following quiver with relations

$$\begin{array}{c} a \\ \curvearrowright \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \\ \curvearrowleft \\ c \end{array} \quad \begin{array}{c} b \\ \curvearrowright \\ \circ \quad \quad \quad \circ \\ \curvearrowleft \\ d \end{array} \quad ba = 0, \quad dc = 0$$

and $\text{Band}(\Lambda)$ is the category of the *band objects* in $D^b(\Lambda - \text{mod})$, i.e. those objects which are invariant under the Auslander–Reiten translation in $D^b(\Lambda - \text{mod})$. Using this result, we describe the Auslander–Reiten quiver of Δ_{nd} .

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2. GENERALITIES ON THE NON-COMMUTATIVE CATEGORY OF SINGULARITIES

Let \mathbb{k} be an algebraically closed field, X be a separated excellent Noetherian scheme over \mathbb{k} such that any coherent sheaf on X is a quotient of a locally free sheaf, and Z be the singular locus of X . Let \mathcal{F}' be a coherent sheaf on X , $\mathcal{F} = \mathcal{O} \oplus \mathcal{F}'$ and $\mathcal{A} := \mathcal{E}nd_X(\mathcal{F})$. Consider the non-commutative ringed space $\mathbb{X} = (X, \mathcal{A})$. Note that \mathcal{F} is a *locally projective* coherent left \mathcal{A} -module. The following result is well-known, see e.g. [7, Theorem 2].

Proposition 2.1. *The functor $\mathbb{F} := \mathcal{F} \overset{\mathbb{L}}{\otimes}_X - : \text{Perf}(X) \rightarrow D^b(\text{Coh}(\mathbb{X}))$ is fully faithful.*

Let $\mathbf{P}(X)$ be the essential image of $\mathbf{Perf}(X)$ under \mathbb{F} . This category can be characterized in the following intrinsic way, see for instance [7, Proposition 2]¹

$$\mathrm{Ob}(\mathbf{P}(X)) = \left\{ \mathcal{H}^\bullet \in \mathrm{Ob}\left(D^b(\mathrm{Coh}(\mathbb{X}))\right) \mid \mathcal{H}_x^\bullet \in \mathrm{Im}\left(\mathrm{Hot}^b(\mathrm{add}(\mathcal{F}_x)) \longrightarrow D^b(\mathcal{A}_x - \mathrm{mod})\right) \right\}.$$

Definition 2.2. In the above notations, the *relative singularity category* $\Delta_X(\mathbb{X})$ is the idempotent completion of the Verdier quotient $D^b(\mathrm{Coh}(\mathbb{X}))/\mathbf{P}(X)$. Recall that according to Balmer and Schlichting [2], $\Delta_X(\mathbb{X})$ has a natural structure of a triangulated category.

Remark 2.3. In case X is an affine scheme, triangulated categories of the form $\Delta_X(\mathbb{X})$ were also considered by Chen [10], Thanhoffer de Völcsey and Van den Bergh [24].

The following result seems to be well-known to experts. However, we were not able to find a reference in the literature and therefore give a proof here.

Lemma 2.4. *Let A be a ring and O be its center. Assume that O is Noetherian of Krull dimension d and A is finitely generated as a left O -module. For any $0 \leq e \leq d$ let $A - \mathrm{mod}^e$ be the full subcategory of left Noetherian A -modules $A - \mathrm{mod}$, whose support over O is at most e -dimensional and $D_e^b(A - \mathrm{mod})$ be the full subcategory of $D^b(A - \mathrm{mod})$ consisting of complexes whose cohomology belongs to $A - \mathrm{mod}^e$. Then the canonical functor*

$$D^b(A - \mathrm{mod}^e) \longrightarrow D_e^b(A - \mathrm{mod})$$

*is an equivalence of triangulated categories.*²

Proof. By [15, Proposition 1.7.11], it is sufficient to show the following

Statement. Let M be an arbitrary object of $A - \mathrm{mod}^e$, N an arbitrary object of $A - \mathrm{mod}$ and $\phi: M \rightarrow N$ an arbitrary injective A -linear map. Then there exists an object K of $A - \mathrm{mod}^e$ and a morphism $\psi: N \rightarrow K$ such that $\psi\phi$ is injective.

Indeed, let $E = E(M)$ be an injective envelope of M and $\theta: M \rightarrow E$ the corresponding embedding. Then there exists a morphism $\alpha: N \rightarrow E$ such that $\alpha\phi = \theta$. Note that $K := \mathrm{Im}(\alpha)$ is a left Noetherian A -module. As in [4, Lemma 3.2.5] one can show that for any $\mathfrak{p} \in \mathrm{Spec}(O)$ we have: $E_{\mathfrak{p}} \cong E(M_{\mathfrak{p}})$. Hence, $K_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin \mathrm{Supp}(M)$. This implies that $\mathrm{kr.dim}(\mathrm{Supp}(K)) \leq e$. \square

Remark 2.5. Lemma 2.4 is no longer true if A is assumed to be just left Noetherian. For example, let \mathfrak{g} be a finite dimensional simple Lie algebra over \mathbb{C} and $U = U(\mathfrak{g})$ its universal enveloping algebra. Then U is left Noetherian, see for instance [20, Section I.7]. By Weyl's complete reducibility theorem, the category $U - \mathrm{mod}^0$ of finite dimensional left U -modules is semi-simple. However, higher extensions between finite dimensional modules do not necessarily vanish in $U - \mathrm{mod}$, see for instance [14]. In particular, the triangulated categories $D^b(U - \mathrm{mod}^0)$ and $D_0^b(U - \mathrm{mod})$ are *not* equivalent.

Globalizing the proof of Lemma 2.4, we get the following result.

¹Although the quoted results were stated in [7] in a weaker form, their proofs can be generalized literally to our case.

²We would like to thank B. Keller and M. Van den Bergh for an enlightening discussion on this subject.

Lemma 2.6. *Let $\mathrm{Coh}_Z(\mathbb{X})$ be the category of coherent left \mathcal{A} -modules, whose support belongs to Z and $D_Z^b(\mathrm{Coh}(\mathbb{X}))$ be the full subcategory of $D^b(\mathrm{Coh}(\mathbb{X}))$ consisting of complexes whose cohomology is supported at Z . Then the canonical functor*

$$D^b(\mathrm{Coh}_Z(\mathbb{X})) \longrightarrow D_Z^b(\mathrm{Coh}(\mathbb{X}))$$

is an equivalence of triangulated categories.

Our next goal is to prove that the category $\Delta_X(\mathbb{X})$ depends only on an open neighborhood of the singular locus Z .

Proposition 2.7. *Let $D_Z^b(\mathrm{Coh}(\mathbb{X}))$ be the full subcategory of $D^b(\mathrm{Coh}(\mathbb{X}))$ consisting of complexes whose cohomology is supported in Z and $\mathrm{P}_Z(X) = \mathrm{P}(X) \cap D_Z^b(\mathrm{Coh}(\mathbb{X}))$. Then the canonical functor*

$$\mathbb{H}: \frac{D_Z^b(\mathrm{Coh}(\mathbb{X}))}{\mathrm{P}_Z(X)} \longrightarrow \frac{D^b(\mathrm{Coh}(\mathbb{X}))}{\mathrm{P}(X)}$$

is fully faithful.

Proof. Our approach is inspired by a recent paper of Orlov [22]. By [15, Proposition 1.6.10], it is sufficient to show that for any $\mathcal{P}^\bullet \in \mathrm{Ob}(\mathrm{P}(X))$, $\mathcal{C}^\bullet \in \mathrm{Ob}(D_Z^b(\mathrm{Coh}(\mathbb{X})))$ and $\varphi: \mathcal{P}^\bullet \rightarrow \mathcal{C}^\bullet$ there exists $\mathcal{Q}^\bullet \in \mathrm{Ob}(\mathrm{P}_Z(X))$ and a factorization

$$\begin{array}{ccc} \mathcal{P}^\bullet & \xrightarrow{\varphi} & \mathcal{C}^\bullet \\ & \searrow \varphi' & \nearrow \varphi'' \\ & \mathcal{Q}^\bullet & \end{array}$$

By Lemma 2.6, we know that the functor $D_Z^b(\mathrm{Coh}(\mathbb{X})) \rightarrow D^b(\mathrm{Coh}_Z(\mathbb{X}))$ is an equivalence of categories. Hence, we may without loss of generality assume that \mathcal{C}^\bullet is a bounded complex of objects of $\mathrm{Coh}_Z(\mathbb{X})$. Let $\mathcal{I} = \mathcal{I}_Z$ be the ideal sheaf of Z . Then there exists $t \geq 1$ such that \mathcal{I}^t annihilates every term of \mathcal{C}^\bullet . Consider the ringed space $\mathbb{Y} = (Z, \mathcal{A}/\mathcal{I}^t)$. Then we have a morphism of ringed spaces $\eta: \mathbb{Y} \rightarrow \mathbb{X}$ and an adjoint pair

$$\begin{cases} \eta_* = \text{forgetf}: & D^-(\mathrm{Coh}(\mathbb{Y})) \rightarrow D^-(\mathrm{Coh}(\mathbb{X})) \\ \eta^* = \mathcal{A}/\mathcal{I}^t \otimes_{\mathcal{A}} - : & D^-(\mathrm{Coh}(\mathbb{X})) \rightarrow D^-(\mathrm{Coh}(\mathbb{Y})). \end{cases}$$

Next, there exists $\mathcal{E}^\bullet \in \mathrm{Ob}(D^b(\mathrm{Coh}(\mathbb{Y})))$ such that $\mathcal{C}^\bullet = \eta_*(\mathcal{E}^\bullet)$. Moreover, we have an isomorphism $\gamma: \mathrm{Hom}_{\mathbb{Y}}(\eta^*\mathcal{P}^\bullet, \mathcal{E}^\bullet) \rightarrow \mathrm{Hom}_{\mathbb{X}}(\mathcal{P}^\bullet, \eta_*(\mathcal{E}^\bullet))$ such that for $\psi \in \mathrm{Hom}_{\mathbb{Y}}(\eta^*\mathcal{P}^\bullet, \mathcal{E}^\bullet)$ the corresponding morphism $\varphi = \gamma(\psi)$ fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{P}^\bullet & \xrightarrow{\xi_{\mathcal{P}^\bullet}} & \eta_*\eta^*\mathcal{P}^\bullet \\ & \searrow \varphi & \swarrow \eta_*(\psi) \\ & \eta_*\mathcal{E}^\bullet & \end{array}$$

where $\xi: \mathbb{1}_{D^-(\mathbb{X})} \rightarrow \eta_*\eta^*$ is the unit of adjunction. Thus, it is sufficient to find a factorization of the morphism $\xi_{\mathcal{P}^\bullet}$ through an object of $\mathrm{P}_Z(X)$.

By definition of $\mathrm{P}(X)$, there exists a bounded complex of locally free \mathcal{O}_X -modules \mathcal{R}^\bullet such that the complexes \mathcal{P}^\bullet and $\mathcal{F} \otimes_X \mathcal{R}^\bullet$ are isomorphic in $D^b(\mathrm{Coh}(\mathbb{X}))$. Note that we

have the following commutative diagram in the category $\text{Com}^b(\mathbb{X})$ of bounded complexes of coherent left \mathcal{A} -modules:

$$\begin{array}{ccc} \mathcal{F} \otimes_X \mathcal{R}^\bullet & \xrightarrow{1 \otimes \theta_{\mathcal{R}^\bullet}} & \mathcal{F} \otimes_X (\mathcal{O}/\mathcal{I}^t \otimes_X \mathcal{R}^\bullet) \\ & \searrow \zeta_{\mathcal{R}^\bullet} & \swarrow \cong \\ & \mathcal{A}/\mathcal{I}^t \otimes_{\mathcal{A}} (\mathcal{F} \otimes_X \mathcal{R}^\bullet) & \end{array}$$

where $\zeta_{\mathcal{R}^\bullet} = \xi_{\mathcal{P}^\bullet}$ in $D^-(\text{Coh}(\mathbb{X}))$ and $\theta_{\mathcal{R}^\bullet}: \mathcal{R}^\bullet \rightarrow \mathcal{O}/\mathcal{I}^t \otimes_X \mathcal{R}^\bullet$ is the canonical map. Since any coherent sheaf on X is a quotient of a locally free sheaf, there exists a bounded complex \mathcal{K}^\bullet of locally free \mathcal{O}_X -modules (Koszul complex of \mathcal{I}^t)

$$\mathcal{K}^\bullet = (0 \rightarrow \mathcal{K}^m \rightarrow \dots \rightarrow \mathcal{K}^1 \rightarrow \mathcal{K}^0 \rightarrow 0)$$

such that

- $\mathcal{K}^0 = \mathcal{O}$ and $\mathcal{H}^0(\mathcal{K}^\bullet) \cong \mathcal{O}/\mathcal{I}^t$,
- $\mathcal{H}^{-i}(\mathcal{K}^\bullet)$ are supported at Z for all $1 \leq i \leq m$.

Hence, we have a factorization of the canonical morphism $\mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}^t$ in the category of complexes $\text{Com}^b(\text{Coh}(X))$: $\mathcal{O}[0] \rightarrow \mathcal{K}^\bullet \rightarrow \mathcal{O}/\mathcal{I}^t[0]$, which induces a factorization

$$\mathcal{R}^\bullet \rightarrow \mathcal{K}^\bullet \otimes_X \mathcal{R}^\bullet \rightarrow \mathcal{O}/\mathcal{I}^t \otimes_X \mathcal{R}^\bullet$$

of the canonical map $\theta_{\mathcal{R}^\bullet}$. Note that the complex $\mathcal{K}^\bullet \otimes_X \mathcal{R}^\bullet$ is perfect and its cohomology is supported at Z . Hence, we get a factorization of the (derived) adjunction unit $\xi_{\mathcal{P}^\bullet}$.

$$\mathcal{P}^\bullet \cong \mathcal{F} \otimes_X^{\mathbb{L}} \mathcal{R}^\bullet \rightarrow \mathcal{Q}^\bullet := \mathcal{F} \otimes_X^{\mathbb{L}} (\mathcal{K}^\bullet \otimes_X^{\mathbb{L}} \mathcal{R}^\bullet) \rightarrow \mathcal{A}/\mathcal{I}^t \otimes_{\mathcal{A}}^{\mathbb{L}} (\mathcal{F} \otimes_X^{\mathbb{L}} \mathcal{R}^\bullet) \cong \mathcal{A}/\mathcal{I}^t \otimes_{\mathcal{A}}^{\mathbb{L}} \mathcal{P}^\bullet$$

we are looking for. This concludes the proof. \square

Theorem 2.8. *In the notations of Proposition 2.7, the induced functor*

$$\mathbb{H}^\omega: \left(\frac{D_Z^b(\text{Coh}(\mathbb{X}))}{\mathcal{P}_Z(X)} \right)^\omega \rightarrow \left(\frac{D^b(\text{Coh}(\mathbb{X}))}{\mathcal{P}(X)} \right)^\omega$$

is an equivalence of triangulated categories.

Proof. Proposition 2.7 implies that the functor \mathbb{H}^ω is fully faithful. Hence, we have to show it is essentially surjective. It suffices to prove the following

Statement. For any $\mathcal{M}^\bullet \in \text{Ob}(D^b(\text{Coh}(\mathbb{X}))/\mathcal{P}(X))$ there exist $\widetilde{\mathcal{M}}^\bullet \in \text{Ob}(D^b(\text{Coh}(\mathbb{X}))/\mathcal{P}(X))$ and $\mathcal{N}^\bullet \in \text{Ob}(D_Z^b(\text{Coh}(\mathbb{X}))/\mathcal{P}_Z(X))$ such that $\mathcal{M}^\bullet \oplus \widetilde{\mathcal{M}}^\bullet \cong \mathbb{H}(\mathcal{N}^\bullet)$.

Note that we have the following diagram of categories and functors

$$\begin{array}{ccc}
\frac{D^b(\mathrm{Coh}(\mathbb{X}))}{D_Z^b(\mathrm{Coh}(\mathbb{X}))} & \xrightarrow{\mathbb{A}} & D^b\left(\frac{\mathrm{Coh}(\mathbb{X})}{\mathrm{Coh}_Z(\mathbb{X})}\right) \\
\uparrow \mathbb{P} & & \downarrow \iota^* \\
D^b(\mathrm{Coh}(\mathbb{X})) & & \\
\uparrow \mathcal{F} \otimes_X^{\mathbb{L}} - & & \\
\mathrm{Perf}(X) & \xrightarrow{j^*} \mathrm{Perf}(U) \xrightarrow{\mathcal{F}|_U \otimes_U^{\mathbb{L}} -} & D^b(\mathrm{Coh}(U)),
\end{array}$$

where both compositions $\mathrm{Perf}(X) \rightarrow D^b(\mathrm{Coh}(U))$ are isomorphic.

- The functor \mathbb{P} is the canonical projection on the Verdier quotient.
- The functor \mathbb{A} is the canonical equivalence of triangulated categories constructed by Miyachi [21, Theorem 3.2].
- For $U = X \setminus Z$ let \mathbb{U} be the ringed space $(U, \mathcal{A}|_U)$. The functor j^* is the canonical restrictions on an open subset. The functor ι^* is an equivalence of triangulated categories induced by a canonical equivalence of abelian categories $\mathrm{Coh}(\mathbb{X})/\mathrm{Coh}_Z(\mathbb{X}) \rightarrow \mathrm{Coh}(\mathbb{U})$.
- Since the coherent sheaf $\mathcal{F}|_U$ is locally free, the functor

$$\mathcal{F}|_U \otimes_U^{\mathbb{L}} - : \mathrm{Perf}(U) = D^b(\mathrm{Coh}(U)) \longrightarrow D^b(\mathrm{Coh}(U))$$

is an equivalence of triangulated categories induced by a Morita-type equivalence $\mathcal{F}|_U \otimes_U - : \mathrm{Coh}(U) \rightarrow \mathrm{Coh}(U)$.

By a result of Thomason and Trobaugh [26, Lemma 5.5.1], for any $\mathcal{S}^\bullet \in \mathrm{Ob}(\mathrm{Perf}(U))$ there exist $\tilde{\mathcal{S}}^\bullet \in \mathrm{Ob}(\mathrm{Perf}(U))$ and $\mathcal{R}^\bullet \in \mathrm{Ob}(\mathrm{Perf}(X))$ such that $j^*\mathcal{R}^\bullet \cong \mathcal{S}^\bullet \oplus \tilde{\mathcal{S}}^\bullet$. Using the fact that \mathbb{A} , ι^* and $\mathcal{F}|_U \otimes_U^{\mathbb{L}} -$ are equivalences of categories, this implies that for any $\mathcal{M}^\bullet \in \mathrm{Ob}(D^b(\mathrm{Coh}(\mathbb{X})))$ there exist $\tilde{\mathcal{M}}^\bullet \in \mathrm{Ob}(D^b(\mathrm{Coh}(\mathbb{X})))$ and $\mathcal{R}^\bullet \in \mathrm{Ob}(\mathrm{Perf}(X))$ such that $\mathcal{P}^\bullet := \mathcal{F} \otimes_X^{\mathbb{L}} \mathcal{R}^\bullet$ is isomorphic to $\mathcal{M}^\bullet \oplus \tilde{\mathcal{M}}^\bullet$ in the Verdier quotient $D^b(\mathrm{Coh}(\mathbb{X}))/D_Z^b(\mathrm{Coh}(\mathbb{X}))$. The last statement is equivalent to the fact that there exists $\mathcal{T}^\bullet \in \mathrm{Ob}(D^b(\mathrm{Coh}(\mathbb{X})))$ and a pair of distinguished triangles

$$\mathcal{C}_\xi^\bullet \longrightarrow \mathcal{T}^\bullet \xrightarrow{\xi} \mathcal{M}^\bullet \oplus \tilde{\mathcal{M}}^\bullet \longrightarrow \mathcal{C}_\xi^\bullet[1] \quad \text{and} \quad \mathcal{C}_\theta^\bullet \longrightarrow \mathcal{T}^\bullet \xrightarrow{\theta} \mathcal{P}^\bullet \longrightarrow \mathcal{C}_\theta^\bullet[1]$$

in $D^b(\mathrm{Coh}(\mathbb{X}))$ such that \mathcal{C}_ξ^\bullet and $\mathcal{C}_\theta^\bullet$ belong to the category $D_Z^b(\mathrm{Coh}(\mathbb{X}))$. Since $\mathcal{C}_\theta^\bullet$ and \mathcal{T}^\bullet are isomorphic in the Verdier quotient $D^b(\mathrm{Coh}(\mathbb{X}))/\mathrm{P}(X)$, we get a distinguished triangle

$$\mathcal{C}_\xi^\bullet \xrightarrow{\alpha} \mathcal{C}_\theta^\bullet \longrightarrow \mathcal{M}^\bullet \oplus \tilde{\mathcal{M}}^\bullet \longrightarrow \mathcal{C}_\xi^\bullet[1]$$

in $D^b(\mathrm{Coh}(\mathbb{X}))/\mathrm{P}(X)$. The functor $\mathbb{H}: D_Z^b(\mathrm{Coh}(\mathbb{X}))/\mathrm{P}_Z(X) \rightarrow D^b(\mathrm{Coh}(\mathbb{X}))/\mathrm{P}(X)$ is fully faithful, see Proposition 2.7. Hence, $\mathcal{M}^\bullet \oplus \tilde{\mathcal{M}}^\bullet$ belongs to the essential image of \mathbb{H} . Thus, the functor \mathbb{H}^ω is essentially surjective, what concludes the proof. \square

From now on, we assume X has only *isolated singularities* and $Z = \text{Sing}(X) = \{x_1, \dots, x_p\}$. For any $1 \leq i \leq p$ we denote $O_i := \mathcal{O}_{x_i}$, \mathfrak{m}_i the maximal ideal in O_i , $A_i = \mathcal{A}_{x_i}$ and $F_i = \mathcal{F}_{x_i}$. Next, we set $\widehat{O}_i = \varprojlim O_i/\mathfrak{m}_i^t O_i$ to be the \mathfrak{m}_i -adic completion of O_i , $\widehat{A}_i := \varprojlim A_i/\mathfrak{m}_i^t A_i$ and $\widehat{F}_i := \varprojlim F_i/\mathfrak{m}_i^t F_i$. Note that $\widehat{A}_i \cong \text{End}_{\widehat{O}_i}(\widehat{F}_i)$. Let $A_i\text{-fdmod}$ denote the category of finite dimensional left A_i -modules. In this case, Lemma 2.6 yields the following statement.

Lemma 2.9. *The canonical functor $\bigvee_{i=1}^p D^b(A_i\text{-fdmod}) \rightarrow D_Z^b(\text{Coh}(\mathbb{X}))$ is an equivalence of triangulated categories. Let P_i be the full subcategory of $D^b(A_i\text{-fdmod})$ consisting of objects admitting a bounded resolution by objects of $\text{add}(F_i)$. Then this functor restricts to an equivalence $\bigvee_{i=1}^p P_i \rightarrow P_Z(X)$.*

Our next aim is to show that the Verdier quotient $D^b(A_i\text{-fdmod})/P_i$ does not change under passing to the completion.

Lemma 2.10. *Let $\text{Perf}_{\text{fd}}(O_i)$ (respectively $\text{Perf}_{\text{fd}}(\widehat{O}_i)$) be the full subcategory of $D^b(O_i\text{-fdmod})$ (respectively $D^b(\widehat{O}_i\text{-fdmod})$) consisting of those complexes which are quasi-isomorphic to a bounded complex of finite rank free O_i - (respectively \widehat{O}_i)-modules. Let $\text{Perf}_{\text{fd}}(O_i) \rightarrow \text{Perf}_{\text{fd}}(\widehat{O}_i)$ and $D^b(A_i\text{-fdmod}) \rightarrow D^b(\widehat{A}_i\text{-fdmod})$ be the exact functors induced by taking the completion. Then they are both equivalences of categories. Moreover, we have a diagram of categories and functors*

$$\begin{array}{ccc} \text{Perf}_{\text{fd}}(O_i) & \xrightarrow{\quad\quad\quad} & \text{Perf}_{\text{fd}}(\widehat{O}_i) \\ \downarrow F_i^{\mathbb{L}} \otimes_{O_i} - & & \downarrow \widehat{F}_i^{\mathbb{L}} \otimes_{\widehat{O}_i} - \\ D^b(A_i\text{-fdmod}) & \xrightarrow{\quad\quad\quad} & D^b(\widehat{A}_i\text{-fdmod}), \end{array}$$

where both compositions $\text{Perf}_{\text{fd}}(O_i) \rightarrow D^b(\widehat{A}_i\text{-fdmod})$ are isomorphic.

Proof. Since the functors $O_i\text{-fdmod} \rightarrow \widehat{O}_i\text{-fdmod}$ and $A_i\text{-fdmod} \rightarrow \widehat{A}_i\text{-fdmod}$ are equivalences of categories, they induce equivalences $D^b(O_i\text{-fdmod}) \rightarrow D^b(\widehat{O}_i\text{-fdmod})$ and $D^b(A_i\text{-fdmod}) \rightarrow D^b(\widehat{A}_i\text{-fdmod})$. In particular, the functor $\text{Perf}_{\text{fd}}(O_i) \rightarrow \text{Perf}_{\text{fd}}(\widehat{O}_i)$ is fully faithful. In order to show it is essentially surjective, it is sufficient to prove that a non-perfect complex can not become perfect after applying the completion functor. Indeed, $X^\bullet \in \text{Ob}(D^b(O_i\text{-mod}))$ is perfect if and only if there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have: $\text{Hom}(X^\bullet, O_i/\mathfrak{m}_i[n]) = 0$. But this property is obviously preserved under the passing to the completion. \square

Corollary 2.11. *Let \widetilde{P}_i (respectively \widehat{P}_i) be the essential image of the triangle functor $\widehat{F}_i^{\mathbb{L}} \otimes_{\widehat{O}_i} - : \text{Perf}(\widehat{O}_i) \rightarrow D^b(\widehat{A}_i\text{-mod})$ (respectively $\widehat{F}_i^{\mathbb{L}} \otimes_{\widehat{O}_i} - : \text{Perf}_{\text{fd}}(\widehat{O}_i) \rightarrow D^b(\widehat{A}_i\text{-fdmod})$). Then we have an equivalence of triangulated categories*

$$\frac{D^b(A_i\text{-fdmod})}{P_i} \xrightarrow{\quad\quad\quad} \frac{D^b(\widehat{A}_i\text{-fdmod})}{\widehat{P}_i}.$$

Going along the same lines as in Theorem 2.8, one can show that the canonical functor

$$\left(\frac{D^b(\widehat{A}_i - \text{fdmod})}{\widehat{P}_i} \right)^\omega \longrightarrow \left(\frac{D^b(\widehat{A}_i - \text{mod})}{\widetilde{P}_i} \right)^\omega$$

is an equivalence. Summing up, we have equivalences of triangulated categories

$$\bigvee_{i=1}^p \left(\frac{D^b(\widehat{A}_i - \text{mod})}{\widetilde{P}_i} \right)^\omega \xleftarrow{\sim} \bigvee_{i=1}^p \left(\frac{D^b(A_i - \text{fdmod})}{P_i} \right)^\omega \xrightarrow{\sim} \left(\frac{D^b(\text{Coh}(\mathbb{X}))}{P(X)} \right)^\omega =: \Delta_X(\mathbb{X}).$$

Now, let Y be a nodal algebraic curve, $Z = \{x_1, \dots, x_p\}$ the singular locus of Y , $\mathcal{I} = \mathcal{I}_Z$ the ideal sheaf of Z and $\mathcal{F} = \mathcal{O} \oplus \mathcal{I}$. Then $\mathcal{A} = \mathcal{E}nd_Y(\mathcal{F})$ is the *Auslander sheaf of orders* introduced in [7]. Let $\mathbb{Y} = (Y, \mathcal{A})$ be the corresponding non-commutative curve. According to [7, Theorem 2], we have: $\text{gl.dim}(\text{Coh}(\mathbb{Y})) = 2$. Thus, \mathbb{Y} is a non-commutative resolution of Y and Corollary 2.11 specializes to the following statement.

Corollary 2.12. *The triangulated category $\Delta_Y(\mathbb{Y})$ splits into a union of p blocks Δ_{nd} , where Δ_{nd} is the “local” contribution of a singular point of Y (see also Section 4 for an explicit description of Δ_{nd}).*

The goal of the subsequent part of this article is to answer the following questions.

- Is the category $\Delta_Y(\mathbb{Y})$ Hom-finite? What are its indecomposable objects?
- What is the Grothendieck group of $\Delta_Y(\mathbb{Y})$?
- Assume E is a plane nodal cubic curve. What is the relation of $\Delta_E(\mathbb{E})$ with the “quiver description” of $D^b(\text{Coh}(\mathbb{E}))$ from [7, Section 7]?

3. ON THE K-THEORY OF THE RELATIVE SINGULARITY CATEGORY $\Delta_X(\mathbb{X})$

Let O be a *complete Gorenstein* local ring and $F = O \oplus F_1 \oplus \dots \oplus F_r \in O - \text{mod}$, where F_1, \dots, F_r are indecomposable and pairwise non-isomorphic and such that O does not belong to $\text{add}(F_1 \oplus \dots \oplus F_r)$. Let $A = \text{End}_O(F)$, $P(O)$ be the essential image of $\text{Perf}(O)$ under the exact embedding $F \overset{\mathbb{L}}{\otimes}_O - : \text{Perf}(O) \rightarrow D^b(A - \text{mod})$ and

$$\Delta_O(A) := \left(\frac{D^b(A - \text{mod})}{P(O)} \right).$$

The following result is well-known to specialists.

Lemma 3.1. *The triangulated category $D^b(O - \text{mod})/\text{Perf}(O)$ is idempotent complete.*

Proof. By a result of Buchweitz [5], we have an equivalence of triangulated categories

$$\frac{D^b(O - \text{mod})}{\text{Perf}(O)} \xrightarrow{\sim} \underline{\text{MCM}}(O),$$

where $\underline{\text{MCM}}(O)$ is the *stable category* of maximal Cohen–Macaulay modules over O . Hence, it suffices to show that $\underline{\text{MCM}}(O)$ is idempotent complete.

Since the ring O is *complete*, the endomorphism algebra of an indecomposable Noetherian O -module is *local*, see [11, Proposition 6.10]. Let M be any object of $\underline{\text{MCM}}(O)$. Then it admits a decomposition $M \cong M_1 \oplus \dots \oplus M_p$, such that the ring $\underline{\text{End}}_O(M_i)$ is local for any $1 \leq i \leq p$ (in other words, $\underline{\text{MCM}}(O)$ is a *local* category). Hence, $\underline{\text{MCM}}(O)$ is idempotent complete, see for example [8, Corollary 13.9]. \square

The main result of this section is the following.

Theorem 3.2. *The category $\Delta_O(A)$ is idempotent complete. Moreover, if $\text{gl.dim}(A) < \infty$ then $K_0(\Delta_O(A)) \cong \mathbb{Z}^r$.*

Proof. First note that we have the following long exact sequences of abelian groups

$$K_0(\text{Perf}(O)) \xrightarrow{\text{can}} K_0(D^b(O - \text{mod})) \longrightarrow K_0(\underline{\text{MCM}}(O)) \longrightarrow 0$$

and $K_0(\text{Perf}(O)) \xrightarrow{\text{can}} K_0(D^b(O - \text{mod})) \rightarrow K_0((\underline{\text{MCM}}(O))^\omega) \rightarrow K_{-1}(\text{Perf}(O)) \rightarrow K_{-1}(D^b(O - \text{mod}))$, where $K_{-1}(\text{Perf}(O))$ and $K_{-1}(D^b(O - \text{mod}))$ denote the negative K-groups of stable Frobenius categories of Schlichting [23, Section 4], associated with the Frobenius pairs $(\text{Com}^b(\text{add}(O)), \text{Com}_{\text{ac}}^b(\text{add}(O)))$ and $(\text{Com}^{-,b}(\text{add}(O)), \text{Com}_{\text{ac}}^-(\text{add}(O)))$ respectively, see [23, Theorem 1]. By [23, Theorem 7] we have $K_{-1}(D^b(O - \text{mod})) = 0$. Since by Lemma 3.1 the stable category $\underline{\text{MCM}}(O)$ is idempotent complete, we also obtain the vanishing $K_{-1}(\text{Perf}(O)) = 0$. In a similar way, we have long exact sequences

$$K_0(\text{P}(O)) \xrightarrow{\text{can}} K_0(D^b(A - \text{mod})) \longrightarrow K_0(\Delta_O(A)) \longrightarrow 0$$

and $K_0(\text{P}(O)) \xrightarrow{\text{can}} K_0(D^b(A - \text{mod})) \rightarrow K_0(\Delta_O(A)^\omega) \rightarrow K_{-1}(\text{P}(O)) \rightarrow 0$. Since the morphism of Frobenius pairs

$$F \otimes_O - : (\text{Com}^b(\text{add}(O)), \text{Com}_{\text{ac}}^b(\text{add}(O))) \longrightarrow (\text{Com}^b(\text{add}(F)), \text{Com}_{\text{ac}}^b(\text{add}(F)))$$

induces an equivalence of the corresponding stable Frobenius categories, [23, Proposition 7] implies that $K_{-1}(\text{P}(O)) = K_{-1}(\text{Perf}(O)) = 0$. Hence, the canonical homomorphism of abelian groups $K_0(\Delta_O(A)) \rightarrow K_0(\Delta_O(A)^\omega)$ is an isomorphism. By a result of Thomason [25, Theorem 2.1], the canonical functor $\Delta_O(A) \rightarrow \Delta_O(A)^\omega$ is an equivalence of triangulated categories, i.e. the triangulated category $\Delta_O(A)$ is idempotent complete.

Since O is a complete ring, A is semi-perfect with $r + 1$ pairwise non-isomorphic indecomposable projective modules. If $\text{gl.dim}(A) < \infty$ then [11, Proposition 16.7] implies that $K_0(D^b(A - \text{mod})) \cong K_0(A - \text{mod}) \cong \mathbb{Z}^{r+1}$. Moreover, the image of the canonical homomorphism $\text{can} : K_0(\text{P}(O)) \rightarrow K_0(D^b(A - \text{mod}))$ is the free abelian group generated by the class of the projective module F . Hence, $K_0(\Delta_O(A)) \cong \text{coker}(\text{can}) \cong \mathbb{Z}^r$. \square

4. DESCRIPTION OF THE CATEGORY Δ_{nd}

In this section A_{nd} denotes the arrow ideal completion of the path algebra of the following quiver with relations \vec{Q}_{nd}

$$(3) \quad - \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} * \begin{array}{c} \xleftarrow{\gamma} \\ \xrightarrow{\delta} \end{array} + \quad \delta\alpha = 0, \quad \beta\gamma = 0.$$

Remark 4.1. Note that $A_{\text{nd}} = \text{End}_{O_{\text{nd}}} (O_{\text{nd}} \oplus \mathbb{k}[[u]] \oplus \mathbb{k}[[v]])$ is the Auslander algebra of the nodal curve singularity $O_{\text{nd}} = \mathbb{k}[[u, v]]/uv$. In particular $\text{gl.dim}(A_{\text{nd}}) = 2$, see [1] or [7, Remark 1].

Our goal is to study the triangulated Verdier quotient category

$$\Delta_{\text{nd}} := \frac{D^b(A_{\text{nd}} - \text{mod})}{\text{Hot}^b(\text{add}(P_*))} \cong \frac{\text{Hot}^b(\text{pro}(A_{\text{nd}}))}{\text{Hot}^b(\text{add}(P_*))} \cong \Delta_{O_{\text{nd}}}(A_{\text{nd}}),$$

where P_* is the indecomposable projective A_{nd} -module corresponding to the vertex $*$. By Theorem 3.2 we know that Δ_{nd} is idempotent complete and $K_0(\Delta_{\text{nd}}) \cong \langle [P_-], [P_+] \rangle \cong \mathbb{Z}^2$.

Definition 4.2. Let $\sigma, \tau \in \{-, +\}$ and $l \in \mathbb{N}$. A *minimal string* $\mathcal{S}_\tau(l)$ is a complex of indecomposable projective A_{nd} -modules

$$\cdots \longrightarrow 0 \longrightarrow P_\sigma \longrightarrow P_* \longrightarrow \cdots \longrightarrow P_* \longrightarrow P_\tau \longrightarrow 0 \longrightarrow \cdots$$

of length $l + 2$ with differentials given by non-trivial paths of minimal possible length and P_τ located in degree 0. Note, that σ is uniquely determined by τ and l :

$$\begin{cases} \sigma = \tau & \text{if } l \text{ is even,} \\ \sigma \neq \tau & \text{if } l \text{ is odd.} \end{cases}$$

Example 4.3. The two complexes depicted below are minimal strings:

$$\begin{aligned} \bullet \mathcal{S}_+(1) &= \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_- \xrightarrow{\cdot\beta} P_* \xrightarrow{\cdot\gamma} P_+ \longrightarrow 0 \longrightarrow \cdots \\ \bullet \mathcal{S}_+(2) &= \cdots \longrightarrow 0 \longrightarrow P_+ \xrightarrow{\cdot\delta} P_* \xrightarrow{\cdot\alpha\beta} P_* \xrightarrow{\cdot\gamma} P_+ \longrightarrow 0 \longrightarrow \cdots \end{aligned}$$

It is interesting to note that the images of minimal strings remain to be *indecomposable* in Δ_{nd} . In order to prove this, we need the following result of Verdier [28, Proposition II.2.3.3], playing a key role in the sequel.

Lemma 4.4. Let \mathcal{T} be a triangulated category and let $\mathcal{U} \subseteq \mathcal{T}$ be a full triangulated subcategory. Let Y be an object in ${}^\perp\mathcal{U} = \{T \in \text{Ob}(\mathcal{T}) \mid \text{Hom}_{\mathcal{T}}(T, \mathcal{U}) = 0\}$ and let

$$\mathbb{P}: \text{Hom}_{\mathcal{T}}(Y, X) \longrightarrow \text{Hom}_{\mathcal{T}/\mathcal{U}}(Y, X)$$

be the map induced by the localization functor. Then \mathbb{P} is bijective for all X in \mathcal{T} .

Of course, there is a dual result for Y in \mathcal{U}^\perp .

Lemma 4.5. Let $\tau \in \{+, -\}$ and $l \in \mathbb{N}$. Then any minimal string $\mathcal{S} = \mathcal{S}_\tau(l)$ belongs to $\text{Ob}({}^\perp\text{Hot}^b(\text{add}(P_*))) \cap \text{Ob}(\text{Hot}^b(\text{add}(P_*))^\perp)$. Moreover, \mathcal{S} is indecomposable in Δ_{nd} .

Proof. First note that $\text{End}_{\text{Hot}^b(\text{pro}(A_{\text{nd}}))}(\mathcal{S}) \cong \mathbb{k}$. In particular, \mathcal{S} is indecomposable in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$. Next, it is easy to check that for all $m \in \mathbb{Z}$

$$\text{Hom}_{\text{Hot}^b(\text{pro}(A_{\text{nd}}))}(P_*[m], \mathcal{S}) = 0 = \text{Hom}_{\text{Hot}^b(\text{pro}(A_{\text{nd}}))}(\mathcal{S}, P_*[m])$$

holds. Now, Verdier's Lemma 4.4 implies indecomposability of \mathcal{S} in Δ_{nd} . \square

Definition 4.6. Let \mathcal{T} be an idempotent complete triangulated category and $X_1, \dots, X_n \in \text{Ob}(\mathcal{T})$ an arbitrary collection of objects. Then $\text{Tria}(X_1, \dots, X_n) \subseteq \mathcal{T}$ is the smallest full triangulated subcategory of \mathcal{T} containing all X_i and closed under taking direct summands.

Remark 4.7. The projective resolutions of the simple A_{nd} -modules S_+ and S_- are

$$0 \rightarrow P_- \xrightarrow{\cdot\beta} P_* \xrightarrow{\cdot\gamma} P_+ \rightarrow S_+ \rightarrow 0 \quad \text{and} \quad 0 \rightarrow P_+ \xrightarrow{\cdot\delta} P_* \xrightarrow{\cdot\alpha} P_- \rightarrow S_- \rightarrow 0.$$

Thus $S_{\pm} \cong \mathcal{S}_{\pm}(1)$ are minimal strings. Let $\rho, \sigma, \tau \in \{-, +\}$ and $l \in \mathbb{N}$. The cone of

$$\begin{array}{ccccccc} \mathcal{S}_{\tau}(l) & & 0 & \longrightarrow & P_{\sigma} & \xrightarrow{\cdot d_3} & P_* \xrightarrow{\cdot d_4} \cdots \xrightarrow{\cdot d_{l+2}} P_* \xrightarrow{\cdot d_{l+3}} P_{\tau} \longrightarrow 0 \\ & & & & \downarrow \text{id} & & \\ \mathcal{S}_{\sigma}(1)[l+1] & & 0 & \longrightarrow & P_{\rho} & \xrightarrow{\cdot d_1} & P_* \xrightarrow{\cdot d_2} P_{\sigma} \longrightarrow 0 \end{array}$$

is isomorphic to the following minimal string

$$\mathcal{S}_{\tau}(l+1)[1] \quad 0 \longrightarrow P_{\rho} \xrightarrow{\cdot d_1} P_* \xrightarrow{\cdot d_2 d_3} \cdots \xrightarrow{\cdot d_{l+2}} P_* \xrightarrow{\cdot d_{l+3}} P_{\tau} \longrightarrow 0.$$

Hence, the minimal strings are generated by S_+ and S_- . In other words, $\mathcal{S}_{\tau}(l)[n]$ is contained in $\text{Tri}(S_+, S_-) \subseteq D^b(A_{\text{nd}} - \text{mod})$, for all $\tau \in \{-, +\}$, $l \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Theorem 4.8. *We use the notations from above.*

(a) *Let X be an indecomposable complex in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$. Then the image of X in Δ_{nd} is either zero or isomorphic to one of the following objects*

$$P_{\sigma}[n] \oplus P_{\tau}[m], \quad P_{\tau}[n] \quad \text{or} \quad \mathcal{S}_{\tau}(l)[n], \quad \text{where } m, n \in \mathbb{Z}, l \in \mathbb{N} \text{ and } \sigma, \tau \in \{+, -\}.$$

(b) *Let $\sigma, \tau \in \{+, -\}$ and $n \in \mathbb{Z}$. We have the following formula:*

$$\text{Hom}_{\Delta_{\text{nd}}}(P_{\sigma}, P_{\tau}[n]) \cong \begin{cases} \mathbb{k} & \text{if } n \in 2\mathbb{Z}_{\leq 0} \text{ and } \sigma = \tau, \\ \mathbb{k} & \text{if } n \in 2\mathbb{Z}_{\leq 0} - 1 \text{ and } \sigma \neq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $P_{\sigma}[n]$ is indecomposable in Δ_{nd} for any $\sigma \in \{+, -\}$ and $n \in \mathbb{Z}$.

(c) *Two objects from the set $\{P_{\sigma}[n], \mathcal{S}_{\tau}(l)[m] \mid \sigma, \tau \in \{+, -\}, n, m \in \mathbb{Z}, l \in \mathbb{N}\}$ are isomorphic in Δ_{nd} if and only if their discrete parameters coincide.*

Proof. Since A_{nd} is a nodal algebra, by the work of Burban and Drozd [6] the indecomposable objects in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$ are explicitly known. They are

- Band objects. These are contained in $\text{Hot}^b(\text{add}(P_*))$ and thus are zero in Δ_{nd} .
- String objects.

The string objects in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$ can be described in the following way. Let $\mathbb{Z}\vec{A}_{\infty}^{\infty}$ be the oriented graph obtained by orienting the edges in a \mathbb{Z}^2 -grid as indicated in Example 4.9 below. Let $\vec{\theta} \subseteq \mathbb{Z}\vec{A}_{\infty}^{\infty}$ be a finite oriented subgraph of type A_n for a certain $n \in \mathbb{N}$. Let Σ and T be the terminal vertices of $\vec{\theta}$ and $\sigma, \tau \in \{-, *, +\}$. We insert the projective modules P_{σ} and P_{τ} at the vertices Σ and T respectively. Next, we plug in P_* at all intermediate vertices of $\vec{\theta}$. Finally, we put maps (given by multiplication with non-trivial paths in \vec{Q}_{nd}) on the arrows between the corresponding indecomposable projective modules. This has to be done in such a way that the composition of two subsequent arrows is always zero. Additionally, at the vertices where $\vec{\theta}$ changes orientation, the inserted paths have to be “alternating”, i.e. if one adjacent path involves α or β then the second should involve γ or

δ . Taking a direct sum of modules and maps in every column of the constructed diagram, we get a complex of projective A_{nd} -modules \mathcal{S} , which we shall simply call *string*.

Example 4.9.

$$\mathcal{S} = \quad \cdots \longrightarrow 0 \longrightarrow P_* \xrightarrow{d_1} \begin{smallmatrix} P_- \\ \oplus \\ P_*^{\oplus 2} \end{smallmatrix} \xrightarrow{d_2} P_* \longrightarrow 0 \longrightarrow \cdots$$

where $d_1 = (0 \quad \cdot(\alpha\beta)^l \quad \cdot(\gamma\delta)^k)^{\text{tr}}$ and $d_2 = (\cdot\beta(\alpha\beta)^n \quad \cdot(\gamma\delta)^m \quad 0)$.

Note, that the strings with $P_\sigma = P_\tau = P_*$ vanish in Δ_{nd} . Therefore, in what follows we may and shall assume that σ or $\tau \in \{-, +\}$.

Proof of (a). Let $\mathcal{S} \in \text{Ob}(\text{Hot}^b(\text{pro}(A_{\text{nd}})))$ be an indecomposable string as defined above.

1. If $P_\tau = P_*$ and $\vec{\theta} = \Sigma \rightarrow \cdots$ hold, then there exists a distinguished triangle

$$\mathcal{S} \xrightarrow{f} P_\sigma[n] \longrightarrow \text{cone}(f) \longrightarrow \mathcal{S}[1]$$

with $\text{cone}(f) \in \text{Ob}(\text{Hot}^b(\text{add}(P_*)))$, yielding an isomorphism $\mathcal{S} \cong P_\sigma[n]$ in Δ_{nd} . Similarly, if $\vec{\theta} = \Sigma \leftarrow \cdots$ holds, then we obtain a triangle $P_\sigma[n] \xrightarrow{f} \mathcal{S} \longrightarrow \text{cone}(f) \longrightarrow P_\sigma[n+1]$ with $\text{cone}(f) \in \text{Ob}(\text{Hot}^b(\text{add}(P_*)))$ and hence an isomorphism $\mathcal{S} \cong P_\sigma[n]$ in Δ_{nd} .

2. We may assume that $\sigma, \tau \in \{-, +\}$. If the graph $\vec{\theta}$ defining \mathcal{S} is *not* linearly oriented (i.e. contains a subgraph $(\star) \circ \longrightarrow \circ \longleftarrow \circ$ or $(\star\star) \circ \longleftarrow \circ \longrightarrow \circ$), then there exists a distinguished triangle of the following form

$$(\star) \quad P_*[s] \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \oplus \mathcal{S}'' \longrightarrow P_*[s+1]$$

$$(\star\star) \quad P_*[s-1] \longrightarrow \mathcal{S}' \oplus \mathcal{S}'' \longrightarrow \mathcal{S} \longrightarrow P_*[s]$$

and therefore $\mathcal{S} \cong \mathcal{S}' \oplus \mathcal{S}'' \cong P_\sigma[n] \oplus P_\tau[m]$ is decomposable in Δ_{nd} .

3. Hence, without loss of generality, we may assume $\sigma, \tau \in \{-, +\}$ and $\vec{\theta}$ to be linearly oriented. If \mathcal{S} has a “non-minimal” differential $d = \cdot p$ (i.e. the path p in \vec{Q}_{nd} contains $\delta\gamma$

or $\beta\alpha$ as a subpath), then we consider the following morphism of complexes

$$\begin{array}{ccccccc} \mathcal{S}' & & P_\sigma & \longrightarrow & P_* & \longrightarrow & \cdots \longrightarrow P_* \\ f \downarrow & & & & & & \downarrow d \\ \mathcal{S}'' & & & & P_* & \longrightarrow & P_* \longrightarrow \cdots \longrightarrow P_* \longrightarrow P_\tau \end{array}$$

which can be completed to a distinguished triangle in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$

$$\mathcal{S}' \xrightarrow{f} \mathcal{S}'' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'[1].$$

By our assumption on d , the morphism f factors through $P_*[s]$ for some $s \in \mathbb{Z}$ and therefore vanishes in Δ_{nd} . Hence, we have a decomposition $\mathcal{S} \cong \mathcal{S}'[1] \oplus \mathcal{S}'' \cong P_\sigma[n] \oplus P_\tau[m]$ in Δ_{nd} .

4. If $\sigma, \tau \in \{-, +\}$, $\vec{\theta}$ is linearly oriented and \mathcal{S} has only minimal differentials, then \mathcal{S} is a minimal string. This concludes the proof of part (a) of Theorem 4.8.

Proof of (b). Every morphism $P_\sigma \rightarrow P_\tau[n]$ in Δ_{nd} is given by a roof $P_\sigma \xleftarrow{f} Q \xrightarrow{g} P_\tau[n]$, where f, g are morphisms in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$ and $\text{cone}(f) \in \text{Ob}(\text{Hot}^b(\text{add}(P_*)))$. By a common abuse of terminology, we call f a *quasi-isomorphism*. Our aim is to find a convenient representative in each equivalence class of roofs. It turns out that $\sigma \in \{+, -\}$ and $n \in \mathbb{Z}$ determine Q and f of our representative and g is either 0 or determined by τ up to scalar.

1. Without loss of generality, we may assume that Q has no direct summands from $\text{Hot}^b(\text{add}(P_*))$. Indeed, if $Q \cong Q' \oplus Q''$ with $Q'' \in \text{Ob}(\text{Hot}^b(\text{add}(P_*)))$, then the diagram

$$\begin{array}{ccccc} & & Q' & & \\ & f' \swarrow & \downarrow \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} & \searrow g' & \\ P_\sigma & \xleftarrow{f=(f' \ f'')} Q' \oplus Q'' & \xrightarrow{g=(g' \ g'')} & P_\tau[n] & \end{array}$$

yields an equivalence of roofs $P_\sigma \xleftarrow{f} Q \xrightarrow{g} P_\tau[n]$ and $P_\sigma \xleftarrow{f'} Q' \xrightarrow{g'} P_\tau[n]$.

2. Using our assumptions on f and Q in conjunction with the description of indecomposable strings in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$, it is not difficult to see that Q can (without restriction) taken to be an indecomposable string with $\tau = *$ and f to be of the following form:

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & P_* & \longrightarrow & P_* \longrightarrow P_* \\ & & & & & & \searrow \\ Q & & & & P_\sigma & \longrightarrow & P_* \longrightarrow \cdots \longrightarrow P_* \longrightarrow P_* \\ f \downarrow & & \text{id} \downarrow & & & & \\ P_\sigma & & P_\sigma & & & & \end{array}$$

3. Without loss of generality, we may assume that Q is constructed from a linearly oriented graph $\vec{\theta}$. Indeed, otherwise we may consider the truncated complex Q^{\leq} defined

in the diagram below and replace our roof by an equivalent one.

$$\begin{array}{c}
 \begin{array}{c} Q \\ \uparrow q \\ Q^{\leq} \end{array} \quad \begin{array}{c} \cdots \longrightarrow P_* \longrightarrow P_* \longrightarrow P_* \\ \searrow \\ P_{\sigma} \xrightarrow{d_1} P_* \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} P_* \xrightarrow{d_n} P_* \\ \text{id} \uparrow \quad \text{id} \uparrow \quad \text{id} \uparrow \quad \text{id} \uparrow \\ P_{\sigma} \xrightarrow{d_1} P_* \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} P_* \xrightarrow{d_n} P_* \end{array} \\
 \\
 \begin{array}{ccc} & Q^{\leq} & \\ f q \swarrow & \downarrow q & \searrow g q \\ P_{\sigma} & Q & P_{\tau}[n] \\ & \xleftarrow{f} & \xrightarrow{g} \end{array}
 \end{array}$$

In particular, $n > 0$ implies that $\text{Hom}_{\Delta_{\text{nd}}}(P_{\sigma}, P_{\tau}[n]) = 0$ holds.

4. By the above reductions, g has the following form:

$$\begin{array}{c} Q \\ \downarrow g \\ P_{\tau}[n] \end{array} \quad \begin{array}{c} P_{\sigma} \longrightarrow P_* \longrightarrow \cdots \longrightarrow P_* \longrightarrow P_* \longrightarrow P_* \longrightarrow \cdots \\ \downarrow g \\ P_{\tau} \end{array}$$

We may truncate again so that Q ends at degree $-n$:

$$\begin{array}{ccc} & Q^{\leq -n} & \\ & \uparrow & \\ P_{\sigma} & Q & P_{\tau}[n] \end{array}$$

5. Next, we may assume that Q has minimal differentials (see Definition 4.2) and thus is uniquely determined by σ and n . Indeed, otherwise there exists a quasi-isomorphism:

$$\begin{array}{c} Q' \\ \downarrow q \\ Q \end{array} \quad \begin{array}{c} P_{\sigma} \longrightarrow P_* \longrightarrow \cdots \longrightarrow P_* \xrightarrow{d'} P_* \longrightarrow 0 \\ \downarrow \text{id} \quad \downarrow \text{id} \quad \downarrow \text{id} \quad \downarrow d'' \quad \downarrow \\ P_{\sigma} \longrightarrow P_* \longrightarrow \cdots \longrightarrow P_* \xrightarrow{d} P_* \longrightarrow P_* \longrightarrow \cdots \end{array}$$

6. Summing up, our initial roof can be replaced by an equivalent one of the following form

$$\begin{array}{ccc} P_{\sigma} & & P_{\sigma} \\ \uparrow f & & \uparrow \text{id} \\ Q & & P_{\sigma} \longrightarrow P_* \longrightarrow \cdots \longrightarrow P_* \\ \downarrow g & & \downarrow g \\ P_{\tau}[n] & & P_{\tau} \end{array}$$

If g is not minimal (i.e. not given by multiplication with a single arrow), then it factors over $P_*[n]$ and therefore vanishes in Δ_{nd} . Thus the morphism space $\text{Hom}_{\Delta_{\text{nd}}}(P_\sigma, P_\tau[n])$ is at most one dimensional. Moreover, g can be non-zero only if n has the right parity.

7. Consider a roof $P_\sigma \xleftarrow{f} Q \xrightarrow{g} P_\tau[n]$ as in the previous step and assume that g is non-zero and minimal. We want to show that the roof defines a non-zero homomorphism in Δ_{nd} . We have a triangle $Q \xrightarrow{g} P_\tau[n] \rightarrow \mathcal{S}_\tau(-n)[n] \rightarrow Q[1]$ in $\text{Hot}^b(A_{\text{nd}} - \text{mod})$ yielding a triangle $P_\sigma \rightarrow P_\tau[n] \rightarrow \mathcal{S}_\tau(-n)[n] \rightarrow P_\sigma[1]$ in Δ_{nd} . Since $\mathcal{S}_\tau(-n)[n]$ is indecomposable, the map is non-zero. The claim follows.

Proof of (c). Note that for $X \in \text{Ob}(\text{Tria}(S_+, S_-))$ we have $[X] = n \cdot ([P_+] + [P_-]) \in K_0(\Delta_{\text{nd}})$ for a certain $n \in \mathbb{Z}$. Thus, the images of indecomposable projective A_{nd} -modules P_+ and P_- are not contained in $\text{Tria}(S_+, S_-)$. By Lemma 4.4 and the classification of indecomposable strings in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$, it remains to show that $P_\sigma[n] \cong P_\tau[m]$ implies $\sigma = \tau$ and $n = m$. Assume that $n > m$ holds. Then using Lemma 4.4 again, we obtain

$$\text{Hom}_{\Delta_{\text{nd}}}(P_\sigma[n], \mathcal{S}_\sigma(1)[n]) \cong \mathbb{k} \neq 0 = \text{Hom}_{\Delta_{\text{nd}}}(P_\tau[m], \mathcal{S}_\sigma(1)[n]).$$

This is a contradiction. Similarly, the assumption $\sigma \neq \tau$ leads to a contradiction. \square

Remark 4.10. Theorem 4.8 and Lemma 4.4 reduce the computation of morphism spaces in Δ_{nd} to a computation in $\text{Hot}^b(\text{pro}(A_{\text{nd}}))$. Moreover, every minimal string may be presented as a cone of a morphism $P_\sigma[n] \rightarrow P_\tau[m]$ in Δ_{nd} (see step 7 in the proof of Theorem 4.8 (b)). Using this fact and the long exact Hom-sequence, one can show that $\dim_{\mathbb{k}} \text{Hom}_{\Delta_{\text{nd}}}(X, Y) \leq 1$ holds for all indecomposable objects X and Y in Δ_{nd} .

Corollary 4.11. *The indecomposable objects of the triangulated subcategory $\text{Tria}(S_-, S_+) \subset D^b(A_{\text{nd}} - \text{mod})$ are precisely the shifts of the minimal strings $\mathcal{S}_\tau(l)$.*

Proof. By Remark 4.7, we know that all minimal strings belong to $\text{Tria}(S_-, S_+)$. Hence, we just have to prove that there are no other indecomposable objects. According to Lemma 4.4 and Lemma 4.5, the functor $\text{Tria}(S_-, S_+) \rightarrow \Delta_{\text{nd}}$ is fully faithful. Therefore, the indecomposable objects of the category $\text{Tria}(S_-, S_+)$ and its essential image in Δ_{nd} are the same. By Theorem 4.8, all indecomposable objects of Δ_{nd} are known and the shifts of the objects P_+ and P_- are not contained in $\text{Tria}(S_-, S_+)$. Hence, the minimal strings are the only indecomposable objects of $\text{Tria}(S_-, S_+)$. \square

5. CONNECTION WITH THE CATEGORY $(D^b(\Lambda - \text{mod})/\text{Band}(\Lambda))^\omega$

Let Λ be the path algebra of the following quiver with relations

$$(4) \quad \begin{array}{c} \begin{array}{ccccc} & a & & b & \\ 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 \\ & c & & d & \end{array} & ba = 0, \quad dc = 0 \end{array}$$

and $\text{Band}(\Lambda)$ be the full subcategory of $D^b(\Lambda - \text{mod})$ consisting of those objects, which are invariant under the Auslander–Reiten translation in $D^b(\Lambda - \text{mod})$. By [7, Corollary 6], the subcategory $\text{Band}(\Lambda)$ is triangulated. Hence, we can define the triangulated category

$$\tilde{\Delta}_{\text{nd}} := (D^b(\Lambda - \text{mod})/\text{Band}(\Lambda))^\omega,$$

i.e. the idempotent completion of the Verdier quotient $D^b(\Lambda - \mathbf{mod})/\mathbf{Band}(\Lambda)$ (see [2]). The main goal of this section is to show that $\tilde{\Delta}_{\text{nd}}$ and Δ_{nd} are triangle equivalent.

Lemma 5.1. *The indecomposable projective Λ -modules are pairwise isomorphic in $\tilde{\Delta}_{\text{nd}}$.*

Proof. Complete the following exact sequences of Λ -modules

$$\begin{aligned} 0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow \left(\begin{array}{ccc} \mathbb{k} & \xrightarrow{1} & \mathbb{k} \\ & \searrow 1 & \nearrow \\ & \mathbb{k} & \xrightarrow{1} 0 \end{array} \right) \longrightarrow 0 \\ 0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow \left(\begin{array}{ccc} 0 & \xrightarrow{\quad} & \mathbb{k} \\ & \searrow & \nearrow 1 \\ & \mathbb{k} & \xrightarrow{1} \mathbb{k} \end{array} \right) \longrightarrow 0 \end{aligned}$$

to triangles in $D^b(\Lambda - \mathbf{mod})$ and note that the modules on the right-hand side are bands. \square

Let $P \in \text{Ob}(\tilde{\Delta}_{\text{nd}})$ be the common image of the indecomposable projective Λ -modules.

Lemma 5.2. *The endomorphisms of P , which are given by the roofs*

$$(5) \quad e_+ = P_1 \xleftarrow{\cdot(a+c)} P_2 \xrightarrow{\cdot a} P_1 \quad \text{and} \quad e_- = P_1 \xleftarrow{\cdot(a+c)} P_2 \xrightarrow{\cdot c} P_1$$

satisfy $e_-e_+ = 0 = e_+e_-$ and $e_- + e_+ = \text{id}_P$ and thus are idempotent. In particular, we have a direct sum decomposition $P \cong P^+ \oplus P^-$, where $P^+ = (P, e_+)$ and $P^- = (P, e_-)$.

Proof. It is clear that $e_- + e_+ = \text{id}_P$. The equality $e_+e_- = 0$ follows from the diagram

$$\begin{array}{ccccc} & & P_3 & & \\ & \swarrow \cdot(b+d) & & \searrow \cdot d & \\ e_+e_- = & P_2 & & P_2 & \\ & \swarrow \cdot(a+c) & \searrow \cdot a & \swarrow \cdot(a+c) & \searrow \cdot c \\ & P_1 & & P_1 & & P_1 \end{array} \quad = \quad P_1 \xleftarrow{\cdot(da+bc)} P_3 \xrightarrow{0} P_1.$$

The second equality $e_-e_+ = 0$ follows from a similar calculation. Hence, $e_{\pm}^2 = e_{\pm}$. \square

Next, note the following easy but useful result.

Lemma 5.3. *Let \mathcal{A} be an abelian category and let $\mathbb{S}: D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ be a triangle equivalence. If $X_1, X_2 \in \text{Ob}(D^b(\mathcal{A}))$ and $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ satisfy*

$$\mathbb{S}^{m_1} X_1 \cong X_1[n_1], \quad \mathbb{S}^{m_2} X_2 \cong X_2[n_2] \quad \text{and} \quad d = m_1 n_2 - m_2 n_1 \neq 0,$$

then $\text{Hom}_{D^b(\mathcal{A})}(X_1, X_2) = 0 = \text{Hom}_{D^b(\mathcal{A})}(X_2, X_1)$.

Proof. By the symmetry of the claim, it suffices to show that $\text{Hom}_{D^b(\mathcal{A})}(X_1, X_2)$ vanishes. Since \mathbb{S} is an equivalence, we have a chain of isomorphisms

$$\begin{aligned} \text{Hom}_{D^b(\mathcal{A})}(X_1, X_2) &\cong \text{Hom}_{D^b(\mathcal{A})}(\mathbb{S}^{\pm m_1 m_2} X_1, \mathbb{S}^{\pm m_1 m_2} X_2) \cong \\ &\text{Hom}_{D^b(\mathcal{A})}(X_1[\pm m_2 n_1], X_2[\pm m_1 n_2]) \cong \text{Hom}_{D^b(\mathcal{A})}(X_1, X_2[\pm d]) \cong \text{Hom}_{D^b(\mathcal{A})}(X_1, X_2[\pm kd]) \end{aligned}$$

for all $k \in \mathbb{N}$. Hence, the claim follows from the boundedness of X_1 and X_2 together with the fact that there are no non-trivial Ext-groups $\text{Ext}_{\mathcal{A}}^{-n}(A_1, A_2) \cong \text{Hom}_{D^b(\mathcal{A})}(A_1, A_2[-n])$, where $A_1, A_2 \in \text{Ob}(\mathcal{A})$ and n is a positive integer. \square

A direct calculation in $D^b(\Lambda - \text{mod})$ yields the following result.

Lemma 5.4. *Let $\mathbb{S}: D^b(\Lambda - \text{mod}) \rightarrow D^b(\Lambda - \text{mod})$ be the Serre functor,*

$$X_+ = \mathbb{k} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{0} \end{array} \mathbb{k} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{1} \end{array} \mathbb{k} \quad \text{and} \quad X_- = \mathbb{k} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{1} \end{array} \mathbb{k} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{0} \end{array} \mathbb{k}.$$

Then $\mathbb{S}(X_{\pm}) \cong X_{\mp}[2]$. In particular, X_{\pm} are $\frac{4}{2}$ -fractionally Calabi–Yau objects.

Corollary 5.5. *The following composition of the inclusion and projection functors*

$$\text{Tria}(X_+, X_-) \hookrightarrow D^b(\Lambda - \text{mod}) \longrightarrow \frac{D^b(\Lambda - \text{mod})}{\text{Band}(\Lambda)}$$

is fully faithful.

Proof. Lemma 5.4 and Lemma 5.3 applied to the Serre functor \mathbb{S} in $D^b(\Lambda - \text{mod})$ imply that $X_{\pm} \in \text{Ob}({}^{\perp}\text{Band}(\Lambda)) \cap \text{Ob}(\text{Band}(\Lambda)^{\perp})$. Hence, the claim follows from Lemma 4.4. \square

Theorem 5.6. *There exists an equivalence of triangulated categories*

$$\mathbb{G}: \frac{D^b(A_{\text{nd}} - \text{mod})}{\text{Hot}^b(\text{add}(P_*))} \longrightarrow \left(\frac{D^b(\Lambda - \text{mod})}{\text{Band}(\Lambda)} \right)^{\omega}.$$

Proof. Let $E = V(zy^2 - x^3 - x^2z) \subset \mathbb{P}^2$ be a nodal cubic curve and $\mathcal{F}' = \mathcal{I}$ be the ideal sheaf of the singular point of E . Let $\mathcal{F} = \mathcal{O} \oplus \mathcal{I}$, $\mathcal{A} = \text{End}_E(\mathcal{F})$ and $\mathbb{E} = (E, \mathcal{A})$. By a result of Burban and Drozd [7, Section 7], there exists a triangle equivalence

$$\mathbb{T}: D^b(\text{Coh}(\mathbb{E})) \longrightarrow D^b(\Lambda - \text{mod})$$

identifying the image of the category $\text{Perf}(E)$ with the category $\text{Band}(\Lambda)$. Moreover, by [7, Proposition 12], the functor \mathbb{T} restricts to an equivalence $\text{Tria}(S_+, S_-) \rightarrow \text{Tria}(X_+, X_-)$. This can be summarized by the following commutative diagram of categories and functors

$$(6) \quad \begin{array}{ccccc} \frac{D^b(A_{\text{nd}} - \text{mod})}{\text{Hot}^b(\text{add}(P_*))} & & & & \\ \uparrow \sim & \searrow \mathbb{G} & & & \\ \left(\frac{D^b(A_{\text{nd}} - \text{fdmod})}{\text{Hot}_{\text{fd}}^b(\text{add}(P_*))} \right)^{\omega} & \xrightarrow{\sim} & \left(\frac{D^b(\text{Coh}(\mathbb{E}))}{\text{P}(E)} \right)^{\omega} & \xrightarrow{\sim} & \left(\frac{D^b(\Lambda - \text{mod})}{\text{Band}(\Lambda)} \right)^{\omega} \\ \uparrow \text{can} & & \uparrow \text{can} & & \uparrow \text{can} \\ D^b(A_{\text{nd}} - \text{fdmod}) & \longrightarrow & D^b(\text{Coh}(\mathbb{E})) & \xrightarrow{\mathbb{T}} & D^b(\Lambda - \text{mod}) \\ & & \uparrow & & \uparrow \\ & & \text{Tria}(S_+, S_-) & \xrightarrow{\sim} & \text{Tria}(X_+, X_-) \end{array}$$

where $\mathbb{G}: \Delta_{\text{nd}} \rightarrow \tilde{\Delta}_{\text{nd}}$ is the induced equivalence of triangulated categories. \square

Lemma 5.7. *The indecomposable objects of the triangulated category $\tilde{\Delta}_{\text{nd}}$ are*

- $P^\pm[n] \cong \mathbb{G}(P_\pm[n])$, $n \in \mathbb{Z}$.
- *The indecomposables of the full subcategory $\text{Tria}(X_+, X_-) \cong \mathbb{G}(\text{Tria}(S_+, S_-))$.*

Proof. Consider the projective resolution of the simple A_{nd} -module S_*

$$0 \longrightarrow P_- \oplus P_+ \xrightarrow{(\cdot\beta \cdot\delta)} P_* \longrightarrow S_* \longrightarrow 0.$$

Completing it to a distinguished triangle yields an isomorphism $S_*[-1] \cong P_+ \oplus P_-$ in Δ_{nd} . In the notations of the diagrams (6) and (4), we have $\mathbb{T}(S_*) \cong P_3[1]$ and therefore

$$\mathbb{G}(P_+ \oplus P_-) \cong \mathbb{G}(S_*[-1]) \cong P \cong (P^+ \oplus P^-).$$

Recall that $X_+ \cong (P_3 \xrightarrow{d} P_2 \xrightarrow{c} P_1)$, where P_1 is located in degree 0 and $P^\pm := (P, e_\pm) \in \text{Ob}(\tilde{\Delta}_{\text{nd}})$, with e_\pm as defined in (5). A direct calculation shows that the obvious morphism from P_1 to X_+ induces a non-zero morphism $P^+ = (P, e_+) \rightarrow X_+$ in $\tilde{\Delta}_{\text{nd}}$, whereas $\text{Hom}_{\tilde{\Delta}_{\text{nd}}}(P^+, X_-) = 0$. Moreover, it was shown in [7] that $\mathbb{T}(S_\pm) \cong X_\pm$. This implies $\mathbb{G}(S_\pm) \cong X_\pm$ and thus $\mathbb{G}(P_\pm) \cong P^\pm$. Theorem 4.8 and Corollary 4.11 yield the stated classification of indecomposables in $\tilde{\Delta}_{\text{nd}}$. \square

6. CONCLUDING REMARKS ON Δ_{nd}

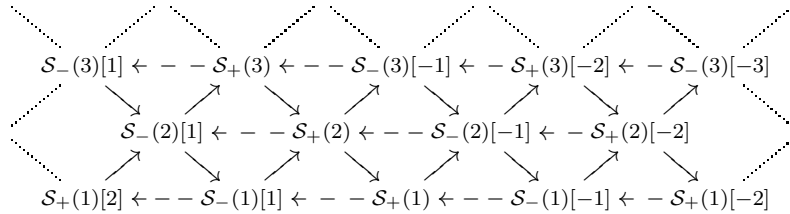
Proposition 6.1. *The category $\text{Tria}(S_+, S_-) \subset \Delta_{\text{nd}}$ has Auslander–Reiten triangles.*

Proof. As mentioned above, we have an exact equivalence of triangulated categories

$$\text{Tria}(S_+, S_-) \cong \text{Tria}(X_+, X_-) \subset D^b(\Lambda - \text{mod}).$$

The category $D^b(\Lambda - \text{mod})$ has a Serre functor \mathbb{S} and therefore has Auslander–Reiten triangles, see [13]. Let $\tau = \mathbb{S} \circ [-1]$ be the Auslander–Reiten translation. Using that τ is an equivalence and Lemma 5.4, we obtain $\tau(\text{Tria}(X_+, X_-)) \cong \text{Tria}(\tau(X_+), \tau(X_-)) \cong \text{Tria}(X_+, X_-)$. Now, the restriction of τ to $\text{Tria}(X_+, X_-)$ is the Auslander–Reiten translation of this subcategory. \square

Remark 6.2. One can show that the Auslander–Reiten quiver of $\text{Tria}(S_+, S_-)$ consists of two $\mathbb{Z}A_\infty$ -components. We draw one of them below, indicating the action of the Auslander–Reiten translation by $\leftarrow - - -$. The other component is obtained from this one by changing the roles of $+$ and $-$.



The category Δ_{nd} does not have Auslander–Reiten triangles, but we may still consider the quiver of irreducible morphisms in Δ_{nd} , which has two *additional* A_∞^∞ -components.

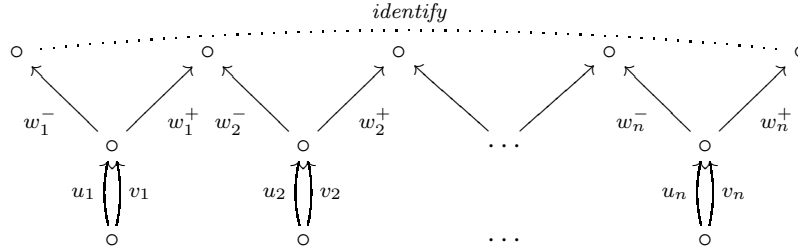
$$\cdots \longrightarrow P_\pm[2] \longrightarrow P_\mp[1] \longrightarrow P_\pm \longrightarrow P_\mp[-1] \longrightarrow P_\pm[-2] \longrightarrow \cdots$$

Proposition 6.3. *The respective triangulated categories $\text{Tria}(X_+, X_-)$ and Δ_{nd} are not triangle equivalent to the bounded derived category of a finite dimensional algebra.*

Proof. Assume that there exists a triangle equivalence to the derived category of a finite dimensional algebra A . Then $D^b(A - \text{mod})$ is of discrete representation type. Hence, A is a *gentle* algebra occurring in Vossieck’s classification [29]. In particular, A is a Gorenstein algebra [12]. Therefore, the Nakayama functor defines a Serre functor on $\text{Hot}^b(\text{pro}(A))$ [13], whose action on objects is described in [3, Theorem B]. On the other hand, $\mathbb{S}^2(X) \cong X[4]$ holds for all objects X in $\text{Tria}(X_+, X_-)$, by Lemma 5.4 and Proposition 6.1. This yields a contradiction. \square

The following proposition generalizes Theorem 5.6.

Proposition 6.4. *Let $n \geq 1$ and Λ_n be the path algebra of the following quiver*



subject to the relations $w_i^- u_i = 0$ and $w_i^+ v_i = 0$ for all $1 \leq i \leq n$. Then

$$\Delta_n := \left(\frac{D^b(\Lambda_n - \text{mod})}{\text{Band}(\Lambda_n)} \right)^\omega \cong \bigvee_{i=1}^n \Delta_{\text{nd}}.$$

In particular, the category Δ_n is representation discrete, Hom-finite and $K_0(\Delta_n) \cong (\mathbb{Z}^2)^{\oplus n}$.

Proof. Let $E = E_n$ be a Kodaira cycle of n projective lines and $\mathbb{E} = (E, \mathcal{E}nd_E(\mathcal{O} \oplus \mathcal{I}_Z))$, where \mathcal{I}_Z is the ideal sheaf of the singular locus Z . By [7, Proposition 10], there exists an equivalence of triangulated categories $D^b(\text{Coh}(\mathbb{E})) \xrightarrow{\sim} D^b(\Lambda_n - \text{mod})$ identifying $\text{Perf}(E) \cong \text{P}(E) \subset D^b(\text{Coh}(\mathbb{E}))$ with $\text{Band}(\Lambda_n) \subset D^b(\Lambda_n - \text{mod})$, see [7, Corollary 6]. Thus, Corollary 2.11 yields the proof. \square

Remark 6.5. Let A be the path algebra of the Kronecker quiver $\circ \begin{smallmatrix} \longrightarrow \\ \longleftarrow \end{smallmatrix} \circ$ and $\text{Band}(A)$ be the full subcategory of $D^b(A - \text{mod})$ consisting of those objects, which are invariant under the Auslander–Reiten translation. Note, that the objects of $\text{Band}(A)$ are direct sums of indecomposable objects lying in tubes. Moreover, $\text{Band}(A)$ is closed under taking cones and direct summands. In particular, it is a triangulated subcategory. It is interesting to note, that the Verdier quotient category $D^b(A - \text{mod})/\text{Band}(A)$ is *not* Hom-finite.

Indeed, the well-known tilting equivalence $D^b(A - \text{mod}) \rightarrow D^b(\text{Coh}(\mathbb{P}^1))$ identifies $\text{Band}(A)$ with the category $D^b(\text{Tor}(\mathbb{P}^1))$, where $\text{Tor}(\mathbb{P}^1)$ is the category of torsion coherent sheaves on \mathbb{P}^1 . Hence, by Miyachi's theorem [21] we have:

$$\frac{D^b(A - \text{mod})}{\text{Band}(A)} \cong \frac{D^b(\text{Coh}(\mathbb{P}^1))}{D^b(\text{Tor}(\mathbb{P}^1))} \cong D^b\left(\frac{\text{Coh}(\mathbb{P}^1)}{\text{Tor}(\mathbb{P}^1)}\right) \cong D^b(\mathbb{k}(t) - \text{mod}),$$

where $\mathbb{k}(t)$ is the field of rational functions. Therefore, the category $D^b(A - \text{mod})/\text{Band}(A)$ is not Hom-finite.

We conclude this paper by giving a relation between our non-commutative singularity category Δ_{nd} and the (classical) singularity category $\underline{\text{MCM}}(O_{\text{nd}})$ for the ring $O_{\text{nd}} = \mathbb{k}[[u, v]]/uv$.

Proposition 6.6. *There is an equivalence of triangulated categories*

$$\frac{\Delta_{\text{nd}}}{\text{Tria}(S_+, S_-)} \xrightarrow{\sim} \underline{\text{MCM}}(O_{\text{nd}}).$$

Proof. The functor $\text{Hom}_{A_{\text{nd}}}(P_*, -): A_{\text{nd}} - \text{mod} \rightarrow O_{\text{nd}} - \text{mod}$ is exact and induces an equivalence of abelian categories $A_{\text{nd}} - \text{mod}/\text{add}(S_+ \oplus S_-) \xrightarrow{\sim} O_{\text{nd}} - \text{mod}$ [7, Theorem 4.8]. Using Miyachi's compatibility of Serre and Verdier quotients [21, Theorem 3.2], we see that $\mathbb{P} = \text{Hom}_{A_{\text{nd}}}(P_*, -): D^b(A_{\text{nd}} - \text{mod}) \rightarrow D^b(O_{\text{nd}} - \text{mod})$ is a quotient functor in the sense of [9], i.e. $D^b(A_{\text{nd}} - \text{mod})/\ker(\mathbb{P}) \xrightarrow{\sim} D^b(O_{\text{nd}} - \text{mod})$. A direct calculation shows that $\text{Hom}_{A_{\text{nd}}}(P_*, P_*) \cong O_{\text{nd}}$. Hence, \mathbb{P} induces a functor $\mathbb{I}: \Delta_{\text{nd}} \rightarrow \underline{\text{MCM}}(O_{\text{nd}})$ and the following diagram commutes.

$$\begin{array}{ccccc} \text{Hot}^b(\text{add}(P_*)) & \hookrightarrow & D^b(A_{\text{nd}} - \text{mod}) & \xrightarrow{\text{can}} & \Delta_{\text{nd}} \\ \downarrow \mathbb{P} & & \downarrow \mathbb{P} & & \downarrow \mathbb{I} \\ \text{Perf}(O_{\text{nd}}) & \hookrightarrow & D^b(O_{\text{nd}} - \text{mod}) & \xrightarrow{\text{can}} & \underline{\text{MCM}}(O_{\text{nd}}) \end{array}$$

By [9, Lemma 2.1], \mathbb{I} is again a quotient functor. Using the classification of indecomposable objects in Δ_{nd} and the fact that $\text{Hom}_{A_{\text{nd}}}(P_*, P_+ \oplus P_-) \cong \mathbb{k}[[u]] \oplus \mathbb{k}[[v]]$, we obtain $\ker(\mathbb{I}) = \text{Tria}(S_+, S_-)$. This concludes the proof. \square

Remark 6.7. After submitting this article, we learned that Thanhoffer de Völcsy and Van den Bergh proved a general Theorem, which contains Proposition 6.6 as a special case. Namely, in the notations of Section 3 assume that O is an *isolated* singularity and A has finite global dimension. Let $e \in A$ be the idempotent corresponding to the identity endomorphism of O . We denote the simple A -modules S_0, \dots, S_r in such a way that S_0 has projective cover Ae . Then the exact functor $eA \otimes_A -: A - \text{mod} \rightarrow O - \text{mod}$ induces an equivalence of triangulated categories

$$(7) \quad \frac{\Delta_O(A)}{\text{Tria}(S_1, \dots, S_r)} \xrightarrow{\sim} \underline{\text{MCM}}(O),$$

see [24, Theorem 5.1.1 and Lemma 5.1.3]. Moreover, they show that $\Delta_O(A)$ is Hom-finite in this generality, see [24, Proposition 5.1.4]. Using quite different methods, slight generalizations of both results were subsequently obtained in recent work of Kalck and Yang [16].

7. SUMMARY

In this section, we collect the major results obtained in this article. Let Y be a nodal algebraic curve, Z its singular locus, $\mathcal{I} = \mathcal{I}_Z$ and $\mathbb{Y} = (Y, \mathcal{A})$ for $\mathcal{A} = \mathcal{E}nd_Y(\mathcal{O} \oplus \mathcal{I})$. Similarly, let $O = \mathbb{k}\llbracket u, v \rrbracket / (uv)$, $\mathfrak{m} = (u, v)$ and $A = \mathbf{End}_O(O \oplus \mathfrak{m})$. Then the following results are true.

- The category $\Delta_Y(\mathbb{Y})$ splits into a union of p blocks Δ_{nd} , where p is the number of singular points of Y and $\Delta_{\text{nd}} = \Delta_O(A)$, see Corollary 2.12.
- The category Δ_{nd} is Hom-finite and representation discrete. In particular, its indecomposable objects and the morphism spaces between them are explicitly known, see Theorem 4.8. Moreover, one can compute its Auslander–Reiten quiver, see Remark 6.2.
- We have: $K_0(\Delta_{\text{nd}}) \cong \mathbb{Z}^2$, see Theorem 3.2.

Moreover, the category Δ_{nd} admits an alternative “quiver description” in terms of representations of a certain gentle algebra Λ , see Section 5.

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